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On Centralizers of Parabolic Subgroups in Coxeter Groups

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Abstract

We describe the structure of the centralizer of an arbitrary parabolic subgroup in any finitely generated Coxeter group.

1 Introduction

Let (W, S) be a Coxeter system such that S is a finite set. A parabolic subgroup W_I of W is the subgroup generated by a subset I of S. In this paper, we determine the structure of the centralizer $C_W(W_I)$ of W_I in W for arbitrary finite S and I. In particular, we do not assume that W is finite.

The structure of $C_W(W_I)$ has been known in certain cases, such as $I = S$ (in this case, $C_W(W_I)$ is the center of W) and $\#I = 1$ (this case is examined by Brink [1]). However, no corresponding general result for an arbitrary I has been known. Our result generalizes these results to the general case.

W has a well known faithful reflection representation in a real vector space with a symmetric bilinear form (which may not be positive definite in general), and the notion of root system in this vector space (see Section 2.1). Using this terminology, we decompose $C_W(W_I)$ (in Section 3) as $W_{I^{\mathrm{iso}}} \times (W_I^{\perp} \rtimes G_I)$, where $W_{I^{\mathrm{iso}}}$ is the parabolic subgroup generated by the elements of I which are isolated in the Coxeter graph of I, $W_I^{\perp}$ is the subgroup of W generated by reflections fixing all simple roots in W_I, and G_I is a certain subgroup defined in that section. We have $W_{I^{\mathrm{iso}}} \simeq (\mathbb{Z}/2\mathbb{Z})^{\# I^{\mathrm{iso}}}$, and by a result by Deodhar [4] or by Dyer [5], $W_I^{\perp}$ is a Coxeter group, whose Coxeter generators and Coxeter relations can be determined if the root system of W is well understood. Our main objective in this paper is to determine the structure of G_I.

To describe G_I, we define a groupoid (see Section 2.2 for the notion of groupoids) H on the set $S^{(N)}$, $N = \#I$, where $S^{(N)} = \{x = (x_1, x_2, \dots x_N) \in S^N \mid x_i \neq x_j \text{ for all } i \neq j\}$, such that its vertex group $H_{x,x}$ is a normal subgroup of G_I whenever $I = \{x_1, x_2, \ldots, x_N\}$, and introduce a graph $\mathcal{G}$, which we call the transition diagram, based on information on the subsets of S of finite type. Note that $W_{I^{\mathrm{iso}}}$, $W_I^{\perp}$, G_I are denoted by $W_{[x]^{\mathrm{iso}}}$, $W_x^{\perp}$, $G_{x,x}$ respectively in the text, by taking such $x \in S^{(N)}$. Then we present H as a quotient groupoid of the fundamental groupoid of $\mathcal{G}$ (which is a free groupoid), and give a method to specify the generators of the kernel of the quotient map in terms of the directed paths of $\mathcal{G}$. As a consequence, a presentation of $H_{x,x}$ is also obtained. (These are done in Section 4.) Moreover, in Section 5, coset representatives of $G_{x,x}/H_{x,x}$ and their products in $G_{x,x}$ are described using the graph $\mathcal{G}$.

Section 6 deals with certain examples; we compute $C_W(W_I)$ for an affine Coxeter group according to the result of previous sections. Further, in that section, we also consider the case of maximal parabolic subgroups; that is, I is a maximal proper subset of S.

Finally, note that all the proofs of the results are omitted in this paper, by reason of space. The detailed proofs will be given in [8].

2 Background material

2.1 Coxeter groups

In this paper, the basic notations and well-known facts about Coxeter groups are based on Humphreys [7]. Here we do not yet assume that S is a finite set.

A pair (W, S) is called a *Coxeter system* (or simply, W is called a *Coxeter group*) if W is a group presented as

$$W = \langle S \mid s^2 = 1 \text{ for all } s \in S, (ss')^{m_{s,s'}} = 1 \text{ for some } s \neq s' \rangle ,$$

where $m_{s,s'}$ denotes the (possibly infinite) order of ss' in W. (In this paper (W, S) always denotes a Coxeter system.) Then the structure of W is described also by its *Coxeter graph* Γ; this is a simple, undirected graph on S which has an edge between s and s' labeled $m_{s,s'}$ if and only if $m_{s,s'} \geqslant 3$. Note that labels $m_{s,s'}$ are usually omitted if $m_{s,s'} = 3$.

(W, S) has a well-known geometric representation (over $\mathbb{R}$) as follows. Let V be a real vector space with basis $\Pi = \{\alpha_s\}_{s \in S}$ having a symmetric bilinear form $\langle\ ,\ \rangle$ determined by $\langle \alpha_s, \alpha_{s'} \rangle = -\cos(\pi/m_{s,s'})$, where π/∞ is interpreted as 0. Then W acts on V by $s \cdot v = v - 2\langle v, \alpha_s \rangle \alpha_s$ for all $s \in S$, and this action is faithful and preserves the bilinear form. The orbit $\Phi = W \cdot \Pi$ is called the *root system* of (W, S), and its elements are called *roots* of (W, S). Obviously, each element of Π is a root; this is called a *simple root*. Further, a root γ is called *positive*, *negative*, denoted by $\gamma > 0$, $\gamma < 0$, if γ is a linear combination of simple roots with all coefficients nonnegative, nonpositive respectively. For $\Psi \subset \Phi$, let Ψ^+, Ψ^- denote the set of all positive, negative roots of Ψ respectively. Then it is well known (see [7]) that $\Phi^- = -\Phi^+$, $\Phi = \Phi^+ \sqcup \Phi^-$ (disjoint union).

Let $\gamma = w \cdot \alpha_s$ be a root with $s \in S$, $w \in W$. Then the *reflection* $s_\gamma = wsw^{-1}$ about γ is determined independently on the choice of w, s, and acts on V by $s_\gamma \cdot v = v - 2\langle v, \gamma \rangle \gamma$.

For $w \in W$, the *length* $\ell(w)$ of w is the minimal number k such that $w = s_1 s_2 \cdots s_k$ for some $s_i \in S$. Then it is also well known that $\ell(w) = \#\Phi_w^+$, where Φ_w^+ denotes the set of all positive roots $\gamma \in \Phi$ such that $w \cdot \gamma < 0$.

For $I \subset S$, let $W_I = \langle I \rangle$ denote the *parabolic subgroup* of W generated by I and let $\Pi_I = \{\alpha_s\}_{s \in I}$, $V_I = \mathrm{span}_{\mathbb{R}} \Pi_I$ and $\Phi_I = W_I \cdot \Pi_I$. Then (W_I, I) is also a Coxeter system with geometric representation V_I, root system Φ_I and simple roots Π_I. Let Γ_I denote the Coxeter graph of (W_I, I). We often say that I is connected instead of that Γ_I is connected. For example, (W, S) is called *irreducible* if S is connected.

For $I \subset S$, we say that I is of *finite type* if W_I is a finite group. Then W_I has the (unique) element $w_0(I)$ with maximal length, called the *longest element* of W_I, if and only if I is of finite type. Further, the following theorem holds:

Theorem 2.1 (see [9]). *If $I \subset S$ is of finite type, then $w_0(I) \cdot \Pi_I = -\Pi_I$.* □

Note that $w_0(I)$ is involutive. Owing to Theorem 2.1, we define a permutation σ_I on I by $w_0(I) \cdot \alpha_s = -\alpha_{\sigma_I(s)}$ for each $s \in I$. Then $\sigma_I(s) = w_0(I) s w_0(I)$ and so σ_I is an involutive automorphism of Γ_I. Figure 1 shows the table of all finite irreducible Coxeter systems (see, for example, [7] for the classification of finite irreducible Coxeter systems) and the action of σ for each Coxeter system.

2.2 Groupoids

In this paper, the basic notations and well-known facts about groupoids are based on Higgins [6] or Brown [2].

A family $X = \{X_{i,j}\}_{i,j \in V(X)}$ of sets $X_{i,j}$ is called a *graph* on vertex set $V(X)$. We write $x \in X$ when $x \in X_{i,j}$ for some i, j. Now a *groupoid*, certain generalization of a group, G is a graph (in the above sense) satisfying the following axioms:

(G1) For $x \in G_{i,j}$ and $y \in G_{j,k}$, the *composition* (or *multiplication*) $xy \in G_{i,k}$ is defined.

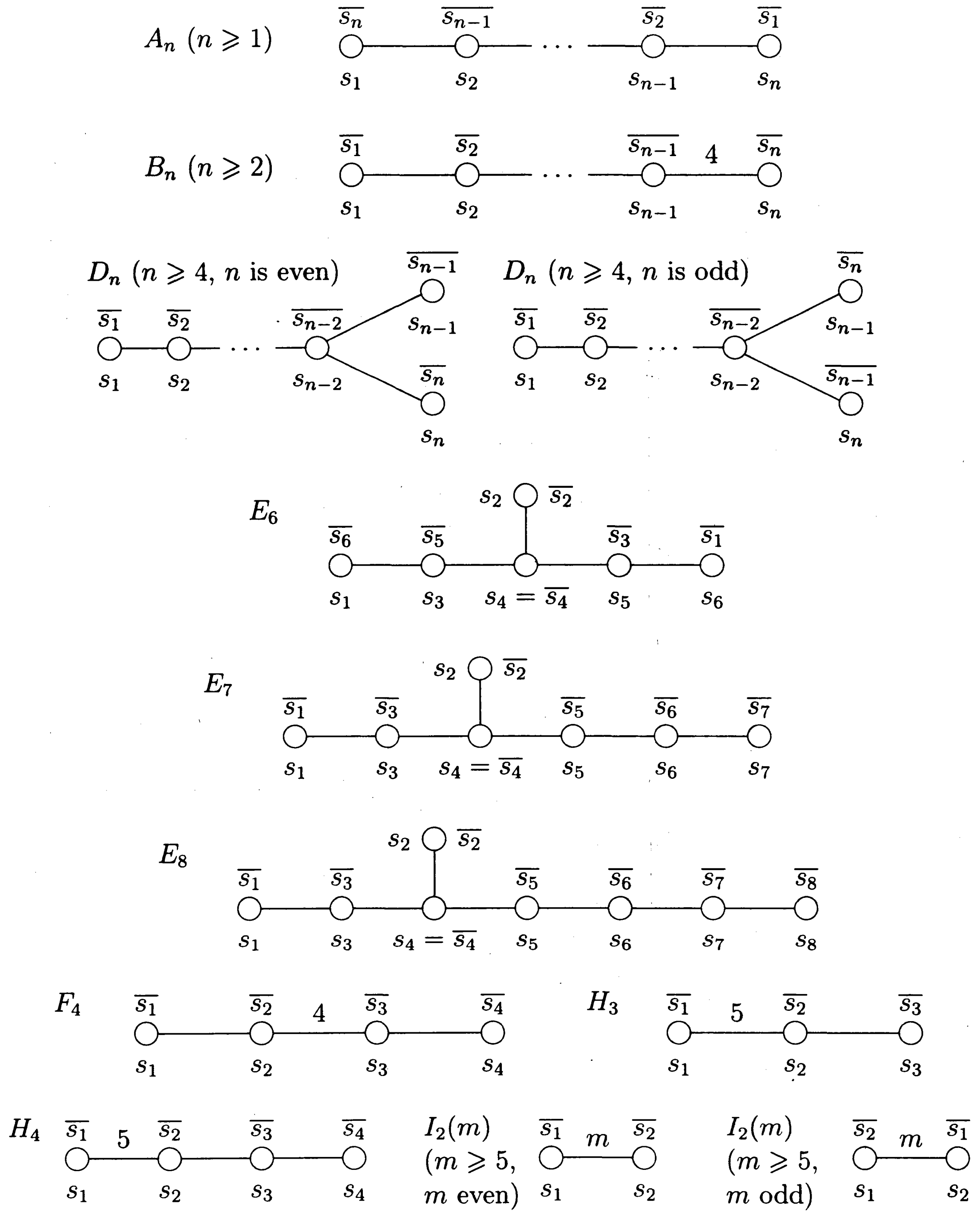
(G2) $(xy)z = x(yz)$ holds for all $x \in G_{i,j}$, $y \in G_{j,k}$, $z \in G_{k,l}$.

(G3) For $i \in V(G)$, there exists a *unit* $1_i \in G_{i,i}$ such that $1_i x = x$ for all $x \in G_{i,j}$, $j \in V(G)$ and $y 1_i = y$ for all $y \in G_{k,i}$, $k \in V(G)$.

(G4) For $x \in G_{i,j}$, there exists an *inverse* $x^{-1} \in G_{j,i}$ of x such that $xx^{-1} = 1_i$ and

Figure 1: Finite irreducible Coxeter systems

Each s_i is the numbering on S, and $\overline{s_i} = \sigma_S(s_i)$.



The unit is unique for each $i \in V(G)$ and x^{-1} is unique for each x, similarly to the case of groups. Note that each $G_{i,i}$, $i \in V(G)$ is a group, called a *vertex group* of G. On the other hand, for a generalization of semigroups, any graph satisfying all axioms above except (G4) is called a *category*. (In the context of the usual category theory, a 'category' defined above is indeed a small category, with *objects* $V(G)$ and *morphisms* $G_{i,j}$. Further, a groupoid is now a small category such that all morphisms are invertible.)

Example 2.2. Now we define the *fundamental groupoid* $\overline{\mathcal{P}} = \overline{\mathcal{P}}(\mathcal{G})$ of any undirected graph $\mathcal{G}$, which is one of the important examples of groupoids.

Let $\mathcal{P}_{i,j} = \mathcal{P}_{i,j}(\mathcal{G})$ be the set of all directed paths of $\mathcal{G}$ from a vertex i to j. Then the family $\{\mathcal{P}_{i,j}\}_{i,j\in V(\mathcal{G})}$, denoted by $\mathcal{P} = \mathcal{P}(\mathcal{G})$, forms a category with concatenation as composition, where the units are trivial paths (paths of length 0) at each vertex.

For each $e \in E(\mathcal{G})$ with certain direction, let e^{-1} denote the same edge but has the opposite direction. Further, for any path $p = e_1e_2\cdots e_n \in \mathcal{P}_{i,j}$, define $p^{-1} = e_n^{-1}\cdots e_2^{-1}e_1^{-1} \in \mathcal{P}_{j,i}$. Now let $\sim$ denote the equivalence relation on $\mathcal{P}_{i,j}$ generated by the relation

$$e_1\cdots e_{k-1}e_ke_k^{-1}e_{k+1}\cdots e_n \sim e_1\cdots e_{k-1}e_{k+1}\cdots e_n,$$

the *homotopy equivalence* of paths. Then the multiplication of $\mathcal{P}$ induces a partial multiplication $[p]\,[q]$ of homotopy classes, such that $[p]\,[q]$ is defined if and only if pq is defined. Let $\overline{\mathcal{P}}_{i,j} = \overline{\mathcal{P}}_{i,j}(\mathcal{G})$ denote the set of homotopy classes of all $p \in \mathcal{P}_{i,j}$. Then the family $\overline{\mathcal{P}} = \{\overline{\mathcal{P}}_{i,j}\}_{i,j\in V(\mathcal{G})}$ forms a groupoid with above multiplication, as required. $\square$

A *subgroupoid* H of a groupoid G is defined similarly to the groups, with the additional condition $V(H) \subset V(G)$. H is called *full* if $H_{i,j} = G_{i,j}$ for all $i, j \in V(H)$, and called *wide* if $V(H) = V(G)$. (Similar notions are also defined for categories and graphs.) Further, H is called *normal* if H is wide and $gxg^{-1} \in H_{j,j}$ for all $x \in H_{i,i}$, $g \in G_{j,i}$.

Example 2.3. For a groupoid G with wide subgroupoid H, let $\sim_H$ be the equivalence relation on $V(G)$ such that $i \sim_H j$ if and only if $H_{i,j} \neq \emptyset$. Then any full subgroupoid of G on an equivalence class with respect to $\sim_G$ is called a *connected component* of G, and G is called *connected* if G consists of only one connected component. $\square$

Let G be a groupoid. Then the *intersection* $\bigcap_\lambda H_\lambda$ of subgroupoids H_λ of G is defined naturally, with $V(\bigcap_\lambda H_\lambda) = \bigcap_\lambda V(H_\lambda)$, and forms a subgroupoid of G. Note that it becomes normal in G whenever all H_λ are. Further, for a subgraph X of G, the *(normal) subgroupoid of G generated by X* is the intersection of all (normal) subgroupoids of G containing X, or equivalently the smallest (normal) subgroupoid of G containing X.

Let X, X' be graphs. A *graph homomorphism* $f : X \to X'$ sends each $i \in V(X)$ to $f(i) \in V(X')$ and each $x \in X_{i,j}$ to $f(x) \in X'_{f(i),f(j)}$. A *graph anti-homomorphism* is defined similarly but $f(x) \in X'_{f(j),f(i)}$, instead of $X'_{f(i),f(j)}$. Let $f : G \to G'$ be a graph homomorphism between groupoids. Then f is called a *groupoid homomorphism* if $f(xy) = f(x)f(y)$ for $x, y \in G$ whenever xy is defined and $f(1_i) = 1_{f(i)}$ for $i \in V(G)$, and *groupoid anti-homomorphisms* are also defined similarly. Then every property about homomorphisms appearing in this subsection can be translated into the case of anti-homomorphisms. Note that $f(x^{-1}) = f(x)^{-1}$ holds for any groupoid homomorphism $f : G \to G'$ and $x \in G$. Further, an *isomorphism* of two groupoids is defined similarly to the case of groups. Note that any groupoid isomorphism $f : G \to G'$ induces isomorphisms of vertex groups $G_{i,i} \to G'_{f(i),f(i)}$.

The *image* of a groupoid homomorphism $f : G \to G'$ is the subgraph $f(G)$ of G' on $f(V(G))$ consisting of all $f(x)$, $x \in G$. Note that $f(G)$ is *not* a subgroupoid of G' in general, but this becomes a subgroupoid whenever f is injective on $V(G)$. On the other hand, the *kernel* of f is the wide subgraph $\ker f$ of G consisting of all $x \in G$ such that $f(x)$ is a unit of G', and then $\ker f$ always forms a normal subgroupoid of G.

For a groupoid G and its normal subgroupoid N, the *quotient groupoid* G/N is defined as follows. Let $V(G/N)$ be the set of all equivalence classes $[i]$ on $V(G)$ with respect to $\sim_N$. For $x, y \in G$, let $\equiv_N$ be an equivalence relation on G such that $x \equiv_N y$ if and only if $x = gyh$ for some $g, h \in N$. Let $[x]$ denote the equivalence class of $x \in G$ with respect to $\equiv_N$. Now define

$$(G/N)_{[i],[j]} = \{[x] \mid x \in G_{i',j'} \text{ for some } i' \in [i], j' \in [j]\}$$

and $G/N = \{(G/N)_{[i],[j]}\}_{[i],[j]}$. Then the multiplication of G/N is induced naturally and G/N forms a groupoid in fact.

Now an analogy of "The First Isomorphism Theorem" is given as follows:

Theorem 2.4 (see [2] or [6]). *If a groupoid homomorphism $f : G \to G'$ is injective on $V(G)$, then the induced map $\overline{f} : G/\ker f \to f(G)$ is an isomorphism.* □

Let G be a groupoid with subgraph X. We say that G is *free on* X if any graph homomorphism $f : X \to G'$ to a groupoid G' extends uniquely to a groupoid homomorphism $\widetilde{f} : G \to G'$. Note that the free groupoid on X is unique (up to isomorphism) if it exists. Conversely, the existence is deduced from the following fact:

Theorem 2.5 (see [6]). *Let $\mathcal{G}$ be an undirected graph. Fix an orientation for $\mathcal{G}$, so $\mathcal{G}$ is considered as a subgraph of its fundamental groupoid $\overline{\mathcal{P}}$. Then $\overline{\mathcal{P}}$ is free on $\mathcal{G}$.* □

3 Decomposition of centralizers

From now on, we assume that S is a finite set. In this section, we state that the centralizer $C_W(W_I)$ of W_I admits a decomposition $W_{[x]^{\mathrm{iso}}} \times (W_x^{\perp} \rtimes G_{x,x})$ as described in Introduction, and define a normal subgroup $H_{x,x}$ of $G_{x,x}$.

We start with some notations. For a nonnegative integer N, let

$$S^{(N)} = \{x = (x_1, \dots, x_N) \in S^N \mid x_i \neq x_j \text{ for all } i \neq j\}$$

and let $[x] = \{x_1, \dots, x_N\}$ for $x \in S^{(N)}$. Further, for $I \subset S$, let I^{iso} be the set of all isolated points of Γ_I and let $I_{\perp} = \{\gamma \in \Phi \mid \langle \gamma, \alpha_s \rangle = 0 \text{ for all } s \in I\}$.

Definition 3.1. For $x, y \in S^{(N)}$, let

$$\begin{aligned} C_{x,y} &= \{w \in W \mid w y_i w^{-1} = x_i \text{ for all } 1 \leqslant i \leqslant N\} \\ &= \{w \in W \mid w \cdot \alpha_{y_i} = \pm \alpha_{x_i} \text{ for all } 1 \leqslant i \leqslant N\}, \\ C'_{x,y} &= \{w \in C_{x,y} \mid w \cdot \alpha_{y_i} = \alpha_{x_i} \text{ for all } y_i \in [y]^{\mathrm{iso}}\}, \\ C''_{x,y} &= \{w \in C_{x,y} \mid w \cdot \alpha_{y_i} = \alpha_{x_i} \text{ for all } 1 \leqslant i \leqslant N\}. \end{aligned}$$ □

Note that $C''_{x,y} \subset C'_{x,y} \subset C_{x,y}$ by this definition. Further, the following lemma also follows from this definition:

Lemma 3.2. *$C = \{C_{x,y}\}_{x,y}$, $C' = \{C'_{x,y}\}_{x,y}$, $C'' = \{C''_{x,y}\}_{x,y}$ are groupoids on $S^{(N)}$.* □

Since the centralizer of each W_I occurs as $C_{x,x}$ by taking $x \in S^{(\#I)}$ such that $[x] = I$, we examine $C_{x,x}$ hereafter. Now we have the decomposition of $C_{x,x}$ as follows:

Theorem 3.3. *Let $x \in S^{(N)}$. Then $C_{x,x} = W_{[x]^{\mathrm{iso}}} \times C'_{x,x}$, $W_{[x]^{\mathrm{iso}}} \simeq (\mathbb{Z}/2\mathbb{Z})^{\#[x]^{\mathrm{iso}}}$.* □

Secondly, we give a certain decomposition of $C'_{x,x}$. Let $W_x^{\perp}$, $x \in S^{(N)}$ be the subgroup of W generated by all reflections s_γ such that $\gamma \in [x]_{\perp}^{+}$, and let

$$G_{x,y} = \{w \in C'_{x,y} \mid \Phi_w^{+} \cap [y]_{\perp} = \emptyset\}$$

for $x, y \in S^{(N)}$. Then it can be shown that $G_{x,y} = \{w \in C'_{x,y} \mid w \cdot [y]_{\perp}^{+} = [x]_{\perp}^{+}\}$, and so $G = \{G_{x,y}\}_{x,y}$ forms a wide subgroupoid of C'. In particular, $G_{x,x}$ is a subgroup of $C'_{x,x}$.

On the other hand, for the subgroup $W_x^{\perp}$, the following lemma holds:

Lemma 3.4. *For $x \in S^{(N)}$, $W_x^{\perp}$ is a normal subgroup of $C'_{x,x}$, and the set $[x]_{\perp}$ is $W_x^{\perp}$-invariant.* □

Deodhar [4] and Dyer [5] proved independently that every reflection subgroup (that is, a subgroup generated by reflections) forms a Coxeter system with certain generating set. Now each $W_x^\perp$ is a reflection subgroup, so $W_x^\perp$ also forms a Coxeter group.

Further, they also gave a characterization of the generating set. Now we determine the generating set $\widetilde{S}_x$ of $W_x^\perp$ by using Deodhar's characterization; let $\widetilde{\Pi}_x$ be the set of all $\gamma \in [x]_\perp^+$ such that γ cannot be written as a nonnegative $\mathbb{R}$-linear combination of other elements of $[x]_\perp^+$, and let $\widetilde{S}_x = \{s_\gamma \mid \gamma \in \widetilde{\Pi}_x\}$. Then we have the following by [4] since $[x]_\perp$ is $W_x^\perp$-invariant (cf. Lemma 3.4):

Theorem 3.5. *$(W_x^\perp, \widetilde{S}_x)$ is a Coxeter system, and its length function $\widetilde{\ell}$ satisfies $\widetilde{\ell}(w) = \#(\Phi_w^+ \cap [x]_\perp)$ for all $w \in W_x^\perp$.* $\square$

We note that Dyer's characterization gives the same generating set with the set obtained by Deodhar's.

Now the decomposition of $C'_{x,x}$ is given as follows:

Theorem 3.6. *$C'_{x,x} = W_x^\perp \rtimes G_{x,x}$ for all $x \in S^{(N)}$.* $\square$

Here we consider the structure of $(W_x^\perp, \widetilde{S}_x)$ and the action of $G_{x,x}$ on $W_x^\perp$ more. Firstly, the following theorem is a special case of Theorem 4.4 of [5]:

Theorem 3.7. *Let $\gamma_1, \gamma_2 \in \widetilde{\Pi}_x$, $\gamma_1 \neq \gamma_2$. Then either $\langle \gamma_1, \gamma_2 \rangle = -\cos(\pi/m)$ for some $m \in \mathbb{Z}$, $m \geqslant 2$ or $\langle \gamma_1, \gamma_2 \rangle \leqslant -1$.* $\square$

Then the structure of $(W_x^\perp, \widetilde{S}_x)$ can be determined whenever $\widetilde{\Pi}_x$ is well understood, by using the following fact:

Proposition 3.8. *Let $\gamma_1, \gamma_2 \in \widetilde{\Pi}_x$, $\gamma_1 \neq \gamma_2$. Then $s_{\gamma_1} s_{\gamma_2}$ has finite order m if and only if $\langle \gamma_1, \gamma_2 \rangle = -\cos(\pi/m)$.* $\square$

Secondly, we examine the action of $G_{x,x}$ on $W_x^\perp$. Let $\widetilde{\Gamma}$ denote the Coxeter graph of $(W_x^\perp, \widetilde{S}_x)$. Note that for arbitrary Coxeter system (W, S) (temporarily we do *not* assume that S is a finite set) with Coxeter graph Γ, each $\sigma \in \mathrm{Aut}\Gamma$ induces an automorphism $f_\sigma : W \to W$, and the map $\mathrm{Aut}\Gamma \to \mathrm{Aut}W$, $\sigma \mapsto f_\sigma$ is a group homomorphism. Now the following theorem follows from Deodhar's characterization of $\widetilde{S}_x$:

Theorem 3.9. *There exists a unique group homomorphism $G_{x,x} \to \mathrm{Aut}\widetilde{\Gamma}$, $w \mapsto \sigma_w$ such that $wuw^{-1} = f_{\sigma_w}(u)$ for all $u \in W_x^\perp$.* $\square$

Corollary 3.10. *$C'_{x,x} = W_x^\perp \times G_{x,x}$ whenever $\mathrm{Aut}\widetilde{\Gamma} = 1$.* $\square$

At last of this section, we define $H = G \cap C''$, so H is a wide subgroupoid of G. Then it follows from the definition that each $H_{x,x}$ is a normal subgroup of $G_{x,x}$. The structures of $G_{x,x}$ and $H_{x,x}$ are discussed in the following sections.

4 Transition diagram and the groupoid H

Before we consider the structure of $G_{x,x}$, we examine the groupoid H in this section.

To do this, we define a graph (which we call the *transition diagram*) $\mathcal{G} = \mathcal{G}^{(N)}(W, S)$ of (W, S) for each nonnegative integer N; it is an undirected graph on $S^{(N)}$ which have close relation with the action of the longest elements of finite parabolic subgroups. Then we construct below a certain anti-homomorphism F from the fundamental groupoid $\overline{\mathcal{P}} = \overline{\mathcal{P}}(\mathcal{G})$ of $\mathcal{G}$ to H which is surjective. This implies that H is anti-isomorphic to the quotient groupoid $\overline{\mathcal{P}}/\ker F$. Finally, we give a certain generating set of $\ker F$ (as a normal subgroupoid) and a method for obtaining a presentation of $H_{x,x}$.

Now we start to define $\mathcal{G}$. For $I \subset S$ and $s \in S$, let $I_{\sim s}$ denote the vertex set of connected component of $\Gamma_{I \cup \{s\}}$ containing s, so $I_{\sim s} \subset I \cup \{s\}$. For $x \in S^{(N)}$, we write $x_{\sim s}$ as a shorthand for $[x]_{\sim s}$. Further, recall (cf. Section 2.1) that we call $I \subset S$ of *finite type* if W_I is a finite group.

Definition 4.1. Let $x \in S^{(N)}$, $s \in S$. Then we say that s *reacts on* x if $s \notin [x]$ and $x_{\sim s}$ is of finite type. In this case, the *product* $\varphi(x, s)$ of this reaction is defined to be the unique element of $S^{(N)}$ such that

$$\varphi(x,s)_i = \begin{cases} \sigma_{x_{\sim s}} \sigma_{x_{\sim s} \smallsetminus \{s\}}(x_i) & \text{if } x_i \in x_{\sim s} \\ x_i & \text{otherwise} \end{cases}$$

and the *residue* of this reaction is $\psi(x, s) = \sigma_{x_{\sim s}}(s)$ (see Section 2.1 for the definition of σ). Moreover, we say that this reaction is *dynamic* if $\varphi(x, s) \neq x$. □

The definition of $\mathcal{G}$ is as follows:

Definition 4.2. Let $x, y \in S^{(N)}$ and $s, s' \in S$. Then $\mathcal{G}$ is defined to be a graph on the vertex set $S^{(N)}$ such that, it has an undirected edge $\{(x, s), (y, s')\}$ between x and y if and only if s reacts dynamically on x and its product, residue are y,s' respectively. □

In addition, when we draw the picture of $\mathcal{G}$, this edge $\{(x, s), (y, s')\}$ is represented, for example, as an edge with labels s close to the vertex x and s' close to y; moreover, for the case $s = s'$, the repeated s's may be replaced by a single s.

Though the definition of edges of $\mathcal{G}$ in Definition 4.2 seems to be asymmetric about (x, s) and (y, s'), $\mathcal{G}$ is well-defined, thanks to the following proposition:

Proposition 4.3. *If s reacts on x, then $\psi(x, s)$ also reacts on $\varphi(x, s)$, and the product, residue of the latter reaction are x,s respectively. In particular, the latter reaction is dynamic if and only if the former is dynamic.* □

This proposition is deduced from the following characterization of reactions:

Lemma 4.4. *Let $x, y \in S^{(N)}$ and $s \in S$. Then s reacts on x and the product is y if and only if the following two conditions hold:*
(i) $[y] \subset [x] \cup \{s\}$,
(ii) *there exists some $w \in W_{[x]\cup\{s\}} \cap C''_{y.x}$, $w \neq 1$.*
Further, $w = w^s_x$ whenever these conditions hold, where $w^s_x = w_0(x_{\sim s})w_0(x_{\sim s} \setminus \{s\})$. □

Moreover, this lemma implies also the following proposition:

Proposition 4.5. *If s reacts on x, then $w^{\psi(x,s)}_{\varphi(x,s)} = (w^s_x)^{-1}$.* □

Example 4.6. Let (W, S) be a finite Coxeter system of type B_5 with numbering on S in Figure 1, and let $N = 3$, $x = (s_1, s_3, s_4) \in S^{(N)}$. Then we show that the connected component $\mathcal{G}_x$ of $\mathcal{G}$ containing x is

$$y \xrightarrow{\;s_3 \quad s_2\;} x \xrightarrow{\quad s_5 \quad} x' \xrightarrow{\;s_2 \quad s_3\;} y'$$

where $x' = (s_1, s_4, s_3)$, $y = (s_4, s_1, s_2)$ and $y' = (s_4, s_2, s_1)$.

Firstly, s_5 reacts on x and $\varphi(x, s_5) = x'$, $\psi(x, s_5) = s_5$. In fact, $x_{\sim s_5} = \{s_3, s_4, s_5\}$, $x_{\sim s_5} \setminus \{s_5\} = \{s_3, s_4\}$ are of type B_3, A_2 respectively. Then we have $\varphi(x, s_5) = x'$, $\psi(x, s_5) = s_5$ since the action of the longest element turns the Coxeter graph for the case of type A_n, while it fixes the Coxeter graph for B_n (cf. Figure 1).

Further, s_2 also reacts dynamically on x,x' and $\varphi(x, s_2) = y$, $\psi(x, s_2) = s_3$, $\varphi(x', s_2) = y'$ and $\psi(x', s_2) = s_3$ by similar argument. Finally, it can be checked that s_5 reacts *not* dynamically on y, y'. Hence the connected component becomes as above. □

For each edge $\{(x, s), (y, s')\}$ of $\mathcal{G}$, let e^s_x denote this edge with direction from x to y (note that y and s' are uniquely determined by x and s whenever the edge exists, so this notation is unambiguous), and let $(e^s_x)^{-1}$ denote the same edge but has the opposite direction (namely, from y to x). Then $(e^s_x)^{-1} = e^{\psi(x,s)}_{\varphi(x,s)}$, and every directed path p of $\mathcal{G}$ is written as the form $p = e^{s_1}_{x_1} e^{s_2}_{x_2} \cdots e^{s_\ell}_{x_\ell}$, with $\varphi(x_i, s_i) = x_{i+1}$ for all $1 \leqslant i \leqslant \ell - 1$. We write $p^{-1} = (e^{s_\ell}_{x_\ell})^{-1} \cdots (e^{s_2}_{x_2})^{-1}(e^{s_1}_{x_1})^{-1}$ for such p. As in Section 2.2, let $\mathcal{P} = \mathcal{P}(\mathcal{G})$, $\mathcal{P}_{x,y} = \mathcal{P}_{x,y}(\mathcal{G})$ denote the set of all directed paths of $\mathcal{G}$, all directed paths of $\mathcal{G}$ from x to y respectively, and let $[p]$ denote the homotopy class of $p \in \mathcal{P}$. Note that $[p^{-1}] = [p]^{-1}$ for any $p \in \mathcal{P}$.

Now we define an anti-homomorphism $F : \overline{\mathcal{P}} \to H$ as follows. F is defined to be the identity map on $S^{(N)}$, and to satisfy $F(e^s_x) = w^s_x$ for each directed edge of $\mathcal{G}$ (here we write $F(e^s_x)$ as a shorthand for $F([e^s_x])$). Then we have $F((e^s_x)^{-1}) = F(e^s_x)^{-1}$ by Proposition 4.5. Since $\overline{\mathcal{P}}$ is a free groupoid on $\mathcal{G}$ (cf. Theorem 2.5), this F extends uniquely to an anti-homomorphism $F : \overline{\mathcal{P}} \to H$ (so $F(e_{x_1}^{s_1} \cdots e_{x_\ell}^{s_\ell}) = w_{x_\ell}^{s_\ell} \cdots w_{x_1}^{s_1}$), provided the following proposition holds:

Proposition 4.7. *If s reacts dynamically on x, then $w_x^s \in H_{\varphi(x,s),x}$.* □

For each $p \in \mathcal{P}$, we also write $F(p)$ as a shorthand for $F([p])$.

The key to the proof of Proposition 4.7 is the following lemma:

Lemma 4.8. *Suppose that s reacts on x. Then $\Phi_{w_x^s}^+ \cap [x]_\perp = \emptyset$ if and only if this reaction is dynamic.* □

Then for each x, s such that s reacts dynamically on x, we have $w_x^s \in C''_{\varphi(x,s),x}$ by definition of w_x^s, while $\Phi_{w_x^s}^+ \cap [x]_\perp = \emptyset$ by this lemma. Hence $w_x^s \in H_{\varphi(x,s),x}$, so Proposition 4.7 holds.

Now we state a theorem which implies that F is surjective, by using the following notations and terminology:

Definition 4.9. For $p = e_{x_1}^{s_1} \cdots e_{x_n}^{s_n} \in \mathcal{P}$, define

$$\ell(p) = n, \ |p| = \sum_{i=1}^{n} \ell(w_{x_i}^{s_i}), \ L(p) = \ell(F(p)) = \ell(w_{x_n}^{s_n} \cdots w_{x_1}^{s_1}).$$

Further, we say that p is *nondegenerate* if $L(p) = |p|$ and *degenerate* if $L(p) < |p|$ (note that $L(p) \leqslant |p|$ for all $p \in \mathcal{P}$). □

The theorem is as follows:

Theorem 4.10. *For each $w \in H_{y,x}$, there exists a nondegenerate path $p \in \mathcal{P}_{x,y}$ such that $F(p) = w$. In particular, F is surjective. Moreover, if $s \in S$ and $w \cdot \alpha_s < 0$, then we can choose such p having e_x^s as the first edge.* □

To prove this theorem, we use the following lemma:

Lemma 4.11. *Let $w \in H_{y,x}$, $s \in S$ and suppose $w \cdot \alpha_s < 0$. Then s reacts dynamically on x and $\ell(w) = \ell(w(w_x^s)^{-1}) + \ell(w_x^s)$.* □

Then Theorem 4.10 follows from this lemma, by induction on $\ell(w)$.

Thus we conclude the construction of the surjective anti-homomorphism F. Now let F_x be the restriction of F to the connected component $\overline{\mathcal{P}}(\mathcal{G}_x)$ of $\overline{\mathcal{P}}$ containing $x \in S^{(N)}$. Then F_x is also a surjective anti-homomorphism from $\overline{\mathcal{P}}(\mathcal{G}_x)$ to the connected component H_x of H containing x. Since F is injective on the vertex set of $\overline{\mathcal{P}}$, F, F_x induce an anti-isomorphism $\overline{\mathcal{P}}/\ker F \to H$, $\overline{\mathcal{P}}(\mathcal{G}_x)/\ker F_x \to H_x$ respectively, as we remarked before.

Finally, we give a generating set of $\ker F_x$ as a normal subgroupoid, and a presentation of $H_{x,x}$. For each $J \subset S$, let $\mathcal{G}_x^{(J)}$ denote the "restriction" of $\mathcal{G}_x$ to J; that is, the subgraph of $\mathcal{G}_x$ consisting of all vertices y of $\mathcal{G}_x$ such that $[y] \subset J$, and all edges $\{(x,s),(y,s')\}$ of $\mathcal{G}_x$ such that $[x],[y] \subset J$ and $s, s' \in J$. Now define $\mathcal{C}_x$ to be the set of all simple closed paths of $\mathcal{G}_x^{(J)}$, where J runs over all subset of S such that $\#J = N+2$ and J is of finite type (actually, for each simple closed path $c = e_{x_1}^{s_1} \cdots e_{x_\ell}^{s_\ell}$ of such $\mathcal{G}_x^{(J)}$, only one of its cyclic permutations $e_{x_i}^{s_i} \cdots e_{x_\ell}^{s_\ell} e_{x_1}^{s_1} \cdots e_{x_{i-1}}^{s_{i-1}}$, $1 \leqslant i \leqslant \ell$, or their inverses must be contained in $\mathcal{C}_x$ and the others may be excluded). Then the following theorem holds, but the proof of this is too long and intricate to write, or even to sketch, in this paper:

Theorem 4.12. $\ker F_x$ *is generated by all* $[c]$, $c \in \mathcal{C}_x$ *as a normal subgroupoid.* $\square$

Further, we consider the presentation of $H_{x,x}$. Let $\mathcal{E}_x$ denote the set of all directed edges of $\mathcal{G}_x$. Then every path of $\mathcal{G}_x$ can be regarded as an element of the free group with basis $\mathcal{E}_x$. Now the following theorem is a special case of Theorem 5.17 of [3]:

Theorem 4.13. *Let T be a maximal tree in $\mathcal{G}_x$. Then the vertex group $(\overline{\mathcal{P}}/\ker F)_{x,x}$ is isomorphic to the group presented by $\langle \mathcal{E}_x \mid \mathcal{C}_x \cup \{ee^{-1} \mid e \in \mathcal{E}_x\} \cup \{e \mid e \in T\}\rangle$.* $\square$

Moreover, the corresponding anti-isomorphism sends each $e_y^s \in \mathcal{E}_x$ to $F(p_{\varphi(y,s)})^{-1} w_y^s F(p_y)$, where p_z denotes the unique reduced path in T from x to z.

5 Representatives of $G_{x,x}/H_{x,x}$ and their product

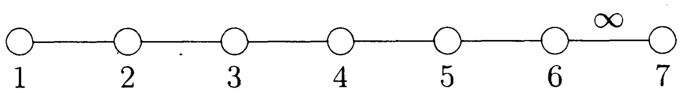
In this section, we examine the quotient group $G_{x,x}/H_{x,x}$. We show below that $G_{x,x}/H_{x,x}$ is a finite elementary abelian 2-group (Corollary 5.5), and that we can choose its coset representatives in the form $w_0(I)F(p)$, where $I \subset [x]$ is of finite type and p is a path of $\mathcal{G}_x$ (Theorem 5.7). Moreover, the multiplication in $G_{x,x}$ is described only by the structure of $\mathcal{G}_x$ (Corollary 5.11); for this description, certain automorphisms on $\mathcal{G}_x$ are defined and used.

We start with some notations. For $x \in S^{(N)}$, define

$$\mathrm{CO}(x) = \{A \subset \{1,2,\ldots,N\} \mid x_A \text{ is a connected component of } x\},$$
$$\mathrm{CO}^{\geq 1}_{<\infty}(x) = \{A \in \mathrm{CO}(x) \mid x_A \text{ is of finite type}, \#A > 1\}$$

where $x_A = \{x_i \mid i \in A\}$. Note that the power set $\mathcal{P}(\mathrm{CO}(x))$ of $\mathrm{CO}(x)$ forms a finite elementary abelian 2-group with symmetric difference as multiplication denoted by $\cdot$

Example 5.1. Let (W,S) be a Coxeter system with Coxeter graph below and let $x = (7,1,3,6,4) \in S^{(5)}$. Then we have $\mathrm{CO}(x) = \{\{1,4\},\{2\},\{3,5\}\}$, $\mathrm{CO}^{\geq 1}_{<\infty}(x) = \{\{3,5\}\}$. $\square$



The following basic lemma is used many times:

Lemma 5.2. *Let $x, y \in S^{(N)}$, $w \in G_{x,y}$.*
(i) $\mathrm{CO}(x) = \mathrm{CO}(y)$ *and* $\mathrm{CO}^{\geq 1}_{<\infty}(x) = \mathrm{CO}^{\geq 1}_{<\infty}(y)$.
(ii) *For $i, j \in A \in \mathrm{CO}(x)$, $w \cdot \alpha_{y_i} = -\alpha_{x_i}$ if and only if $w \cdot \alpha_{y_j} = -\alpha_{x_j}$.*
(iii) *If $i \in A \in \mathrm{CO}(x) \smallsetminus \mathrm{CO}^{\geq 1}_{<\infty}(x)$, then $w \cdot \alpha_{y_i} = \alpha_{x_i}$.* □

For $x, y \in S^{(N)}$ and $\mathcal{A} \subset \mathrm{CO}^{\geq 1}_{<\infty}(y)$, define

$$G^{\mathcal{A}}_{x,y} = \{w \in G_{x,y} \mid w \cdot \alpha_{y_i} = -\alpha_{x_i} \text{ if and only if } i \in \bigcup \mathcal{A}\}.$$

Then Lemma 5.2 yields the following decomposition of $G_{x,y}$:

Lemma 5.3. $G_{x,y} = \bigsqcup_{\mathcal{A} \subset \mathrm{CO}^{\geq 1}_{<\infty}(y)} G^{\mathcal{A}}_{x,y}$ *for all $x, y \in S^{(N)}$.* □

Further, the following lemma is deduced immediately from the definition:

Lemma 5.4. *Let $x, y, z \in S^{(N)}$, $\mathcal{A}, \mathcal{A}' \subset \mathrm{CO}^{\geq 1}_{<\infty}(x)$ and suppose $G_{x,y} \neq \emptyset$, $G_{y,z} \neq \emptyset$ (so all $\mathrm{CO}^{\geq 1}_{<\infty}(x), \mathrm{CO}^{\geq 1}_{<\infty}(y), \mathrm{CO}^{\geq 1}_{<\infty}(z)$ coincide by Lemma 5.2 (i)). Then*

$$G^{\mathcal{A}}_{x,y} \cdot G^{\mathcal{A}'}_{y,z} \subset G^{\mathcal{A} \cdot \mathcal{A}'}_{x,z}, (G^{\mathcal{A}}_{x,y})^{-1} = G^{\mathcal{A}}_{y,x}, G^{\emptyset}_{x,y} = H_{x,y}.$$ □

For $x \in S^{(N)}$, define $E_x = \{\mathcal{A} \subset \mathrm{CO}^{\geq 1}_{<\infty}(x) \mid G^{\mathcal{A}}_{x,x} \neq \emptyset\}$. Then the preceding lemmas imply the structure of $G_{x,x}/H_{x,x}$ as follows:

Corollary 5.5. *Let $x \in S^{(N)}$. Then E_x is a subgroup of $\mathcal{P}(\mathrm{CO}^{\geq 1}_{<\infty}(x))$ and isomorphic to $G_{x,x}/H_{x,x}$, so it is also a finite elementary abelian 2-group. Further, this isomorphism sends each coset $wH_{x,x}$ to the unique $\mathcal{A}_w \subset \mathrm{CO}^{\geq 1}_{<\infty}(x)$ satisfying $w \in G^{\mathcal{A}_w}_{x,x}$.* □

Now we give certain coset representatives of $G_{x,x}/H_{x,x}$. For $x \in S^{(N)}$ and $\mathcal{A} \subset \mathrm{CO}^{\geq 1}_{<\infty}(x)$, define

$$w_0(\mathcal{A}; x) = \prod_{A \in \mathcal{A}} w_0(x_A) = w_0(x_{\bigcup \mathcal{A}})$$

(note that all $w_0(x_A)$ in the above product commute). Further, let $y \in S^{(N)}$, $y \in \mathcal{G}_x$. Then we have $G_{y,x} \neq \emptyset$ since $\mathcal{G}_x$ is connected and $F(\overline{\mathcal{P}}_{x,y}) \subset G_{y,x}$, and so $\mathcal{A} \subset \mathrm{CO}^{\geq 1}_{<\infty}(y)$ by Lemma 5.2 (i). Now define $y^{\mathcal{A}} \in S^{(N)}$ by

$$(y^{\mathcal{A}})_i = w_0(\mathcal{A}; y) y_i w_0(\mathcal{A}; y) = \begin{cases} \sigma_{y_A}(y_i) & \text{if } i \in A \text{ for some } A \in \mathcal{A} \\ y_i & \text{otherwise.} \end{cases}$$

Then we have the following lemma:

Lemma 5.6. *Let $x \in S^{(N)}$, $\mathcal{A} \subset \mathrm{CO}^{\geq 1}_{<\infty}(x)$. Then $w_0(\mathcal{A};x) \in G^{\mathcal{A}}_{x^{\mathcal{A}},x}$. Hence $G^{\mathcal{A}}_{x,x} = w_0(\mathcal{A};x)H_{x^{\mathcal{A}},x}$ and so $E_x = \{\mathcal{A} \subset \mathrm{CO}^{\geq 1}_{<\infty}(x) \mid H_{x^{\mathcal{A}},x} \neq \emptyset\}$.* □

The coset representatives of $G_{x,x}/H_{x,x}$ are given as follows (recall that the map F is surjective):

Theorem 5.7. *$G_{x,x} = \bigsqcup_{\mathcal{A} \in E_x} w_0(\mathcal{A};x)F(p_{\mathcal{A}})H_{x,x}$, where $p_{\mathcal{A}}$ is an arbitrarily chosen element of $\mathcal{P}_{x,x^{\mathcal{A}}}$ for each $\mathcal{A} \in E_x$.* □

Since $H_{x,x}$ is generated by certain elements $F(p)$, $p \in \mathcal{P}_{x,x}$ (cf. Section 4), this theorem implies that $G_{x,x}$ is generated by such $F(p)$ and these coset representatives. In the rest of this section, we describe the multiplication of these generators of $G_{x,x}$, using certain automorphisms on $\mathcal{G}_x$ defined in Theorem 5.10 below.

We use the following two lemmas:

Lemma 5.8. *Let $x \in S^{(N)}$, $\mathcal{A}, \mathcal{A}' \subset \mathrm{CO}^{\geq 1}_{<\infty}(x)$ (so $\mathrm{CO}^{\geq 1}_{<\infty}(x^{\mathcal{A}}) = \mathrm{CO}^{\geq 1}_{<\infty}(x)$ by Lemmas 5.2 (i) and 5.6). Then*

$$w_0(\mathcal{A};x)w_0(\mathcal{A}';x) = w_0(\mathcal{A}\cdot\mathcal{A}';x), w_0(\mathcal{A}';x^{\mathcal{A}}) = w_0(\mathcal{A}';x), (x^{\mathcal{A}})^{\mathcal{A}'} = x^{\mathcal{A}\cdot\mathcal{A}'}.$$ □

Lemma 5.9. *Let $x \in S^{(N)}$, $y, z \in \mathcal{G}_x$, $\mathcal{A} \subset \mathrm{CO}^{\geq 1}_{<\infty}(x)$ and $s \in S$. Then s reacts dynamically on y and $\varphi(y,s) = z$ if and only if s reacts dynamically on $y^{\mathcal{A}}$ and $\varphi(y^{\mathcal{A}}, s) = z^{\mathcal{A}}$. Moreover, if above conditions hold, then $\psi(y,s)$ and $\psi(y^{\mathcal{A}}, s)$ coincide, and $w^s_{y^{\mathcal{A}}} = w_0(\mathcal{A};z)w^s_y w_0(\mathcal{A};y)$.* □

The automorphisms on $\mathcal{G}_x$ are given as follows:

Theorem 5.10. *For each $\mathcal{A} \in E_x$, define $\rho_{\mathcal{A}} : \mathcal{G}_x \to \mathcal{G}_x$ by*

$$\rho_{\mathcal{A}}(y) = y^{\mathcal{A}} \ (y \in V(\mathcal{G}_x)),\ \rho_{\mathcal{A}}(e^s_y) = e^s_{y^{\mathcal{A}}} \ (e^s_y \in E(\mathcal{G}_x)).$$

(i) *$\rho_{\mathcal{A}}$ is an involutive graph automorphism on $\mathcal{G}_x$.*
(ii) *$\rho_{\mathcal{A}}\rho_{\mathcal{A}'} = \rho_{\mathcal{A}\cdot\mathcal{A}'}$ holds for all $\mathcal{A}, \mathcal{A}' \in E_x$.*
(iii) *If $\rho_{\mathcal{A}}$ also denotes the extension of $\rho_{\mathcal{A}}$ to $\mathcal{P}(\mathcal{G}_x)$ (the directed paths of $\mathcal{G}_x$), then it is an involutive automorphism and satisfies $\rho_{\mathcal{A}}(p^{-1}) = \rho_{\mathcal{A}}(p)^{-1}$, $\rho_{\mathcal{A}}\rho_{\mathcal{A}'} = \rho_{\mathcal{A}\cdot\mathcal{A}'}$ and*

$$F(\rho_{\mathcal{A}}(p)) = w_0(\mathcal{A};z)F(p)w_0(\mathcal{A};y)$$

for all $p \in \mathcal{P}(\mathcal{G}_x)_{y,z}$. □

Now the multiplication (in $G_{x,x}$) of the representatives of $G_{x,x}/H_{x,x}$ and the action of these to the generators $F(p)$ of $H_{x,x}$ are described as follows, by using the automorphisms

Corollary 5.11. (i) *Let* $\mathcal{A} \in E_x$, $p_{\mathcal{A}} \in \mathcal{P}_{x,x^{\mathcal{A}}}$ *and* $q \in \mathcal{P}_{x,x}$. *Then*

$$w_0(\mathcal{A};x)F(p_{\mathcal{A}})F(q)\left(w_0(\mathcal{A};x)F(p_{\mathcal{A}})\right)^{-1} = F(\rho_{\mathcal{A}}((p_{\mathcal{A}})^{-1}qp_{\mathcal{A}})).$$

(ii) *Let* $\mathcal{A}, \mathcal{A}' \in E_x$, $p_{\mathcal{A}} \in \mathcal{P}_{x,x^{\mathcal{A}}}$ *and* $p_{\mathcal{A}'} \in \mathcal{P}_{x,x^{\mathcal{A}'}}$. *Then*

$$w_0(\mathcal{A};x)F(p_{\mathcal{A}})w_0(\mathcal{A}';x)F(p_{\mathcal{A}'}) = w_0(\mathcal{A}\cdot\mathcal{A}';x)F(p_{\mathcal{A}'}\rho_{\mathcal{A}'}(p_{\mathcal{A}})). \qquad \square$$

Note that to obtain $\rho_{\mathcal{A}}(p)$ for each $\mathcal{A} \in E_x$ and $p = e_{x_1}^{s_1}\cdots e_{x_n}^{s_n} \in \mathcal{P}(\mathcal{G}_x)$, we need *not* compute $(x_i)^{\mathcal{A}}$ for any $i \geqslant 2$; indeed, we have only to compute $(x_1)^{\mathcal{A}}$, and then start at $(x_1)^{\mathcal{A}}$ and trace each (unique) directed edge labeled s_i step by step.

6 Examples

Example 6.1. (W,S) is of type $\widetilde{B_7}$ and $x = (1,2,4,5,8) \in S^{(N)}$ $(N=5)$, as in Figure 2 (in this section we write i as a shorthand for s_i). Then we compute the centralizer $C_{x,x}$ of $W_{\{1,2,4,5,8\}}$.

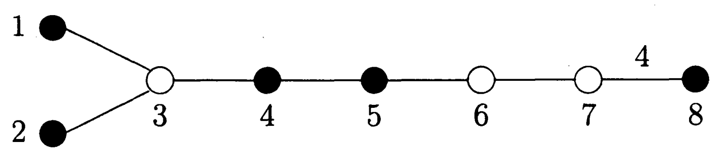


Figure 2: Coxeter graph of type $\widetilde{B_7}$

1. Figure 2 implies $[x]^{\mathrm{iso}} = \{1,2,8\}$, so by Theorem 3.3,

$$C_{x,x} = W_{\{1,2,8\}} \times C'_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^3 \times C'_{x,x}.$$

2. We determine the structure of the Coxeter system $(W_x^{\perp}, \widetilde{S}_x)$. Let

$$\delta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \sqrt{2}\,\alpha_8,$$

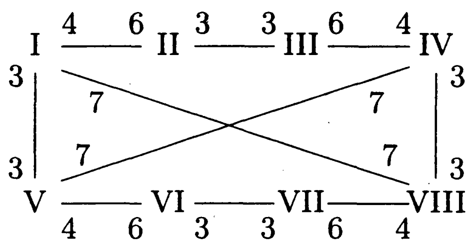
which is called the null root of (W,S). So $\langle \delta, \alpha_i \rangle = 0$ for all $1 \leqslant i \leqslant 8$. Now Φ is the (disjoint) union of following two sets

$$\Phi' = \{n\delta \pm \gamma \mid n \in \mathbb{Z}, \gamma \in \Phi^+_{S\smallsetminus\{8\}}\},$$
$$\Phi'' = \{n\sqrt{2}\,\delta \pm (\sum_{i=k}^{7}\sqrt{2}\,\alpha_i + \alpha_8) \mid n \in \mathbb{Z}, 2 \leqslant k \leqslant 8\}.$$

Moreover, $\Phi' = W \cdot \alpha_i$ for each $1 \leqslant i \leqslant 7$ and $\Phi'' = W \cdot \alpha_8$. To show these, we have only to check that every element of Φ', Φ'' is indeed a root of (W, S) (this can be proved by the induction on $|n|$), both Φ', Φ'' are W-invariant (this follows from that $W \cdot \Phi^+_{S \smallsetminus \{8\}} \subset \Phi'$ and $W \cdot (\sum_{i=k}^{7} \sqrt{2}\,\alpha_i + \alpha_8) \subset \Phi''$ for all $2 \leqslant k \leqslant 8$), and $\Pi_{S \smallsetminus \{8\}} \subset \Phi'$, $\alpha_8 \in \Phi''$ (these are trivial).

By the above result, we have $[x]_\perp = \{n\sqrt{2}\,\delta \pm \beta \mid n \in \mathbb{Z}\}$ and so $\widetilde{\Pi}_x = \{\beta, \beta'\}$, where $\beta = \sqrt{2}\,\alpha_7 + \alpha_8$, $\beta' = \sqrt{2}\,\delta - \beta$. Further, since $\langle \beta, \beta' \rangle = -1$, Proposition 3.8 implies that $s_\beta s_{\beta'}$ has infinite order. Hence $(W_x^\perp, \widetilde{S}_x)$ is of type $\widetilde{A_1}$ (the infinite dihedral group), and by Theorem 3.6, we have $C'_{x,x} = W_x^\perp \rtimes G_{x,x} \simeq \widetilde{A_1} \rtimes G_{x,x}$.

3. The connected component $\mathcal{G}_x$ of $\mathcal{G}$ containing x is as in Figure 3. In this case, let



$\mathrm{I} = (1,2,5,6,8)$
$\mathrm{II} = (1,2,4,5,8) = x$
$\mathrm{III} = (2,1,5,4,8)$
$\mathrm{IV} = (2,1,6,5,8)$
$\mathrm{V} = (2,1,5,6,8)$
$\mathrm{VI} = (2,1,4,5,8)$
$\mathrm{VII} = (1,2,5,4,8) = x^{A_0}$
$\mathrm{VIII} = (1,2,6,5,8)$

Figure 3: Connected component of $\mathcal{G}$

$e(y, z)$ denote the unique directed edge of $\mathcal{G}_x$ from y to z. Now we determine the structure of the groupoid H, by using Theorems 4.12 and 4.13. Firstly, we examine the generating set of $\ker F_x$. Since $N + 2 = 7 = \#S - 1$, we have only to consider $\mathcal{G}_x^{(J)}$ for $J = S \smallsetminus \{s\}$, $s \in S$. For example, if $s = 4$, then we obtain $\mathcal{G}_x^{(J)}$ from $\mathcal{G}_x$ by deleting four vertices II, III, VI, VII and six edges $e(\mathrm{I},\mathrm{II}), e(\mathrm{II},\mathrm{III}), e(\mathrm{III},\mathrm{IV}), e(\mathrm{V},\mathrm{VI}), e(\mathrm{VI},\mathrm{VII}), e(\mathrm{VII},\mathrm{VIII})$. By similar argument, $\mathcal{G}_x^{(J)}$ is nonempty for $s = 3, 4, 6, 7$, as in Figure 4, while this is empty for $s = 1, 2, 5, 8$. Now by Theorem 4.12, $\ker F_x$ is generated (as a normal subgroupoid) by $[c_1]$ and $[c_2]$, where

$$c_1 = e(\mathrm{I},\mathrm{VIII})e(\mathrm{VIII},\mathrm{IV})e(\mathrm{IV},\mathrm{V})e(\mathrm{V},\mathrm{I}),$$
$$c_2 = e(\mathrm{I},\mathrm{II})e(\mathrm{II},\mathrm{III})e(\mathrm{III},\mathrm{IV})e(\mathrm{IV},\mathrm{VIII})e(\mathrm{VIII},\mathrm{VII})e(\mathrm{VII},\mathrm{VI})e(\mathrm{VI},\mathrm{V})e(\mathrm{V},\mathrm{I})$$

(note that in this case, every proper subset of S is of finite type).

Secondly, we give a presentation of $H_{x,x}$ by Theorem 4.13. Recall that $\mathcal{E}_x$ denotes the set of all directed edges of $\mathcal{G}_x$. Now we choose a maximal tree T in $\mathcal{G}_x$ as in Figure 5, then

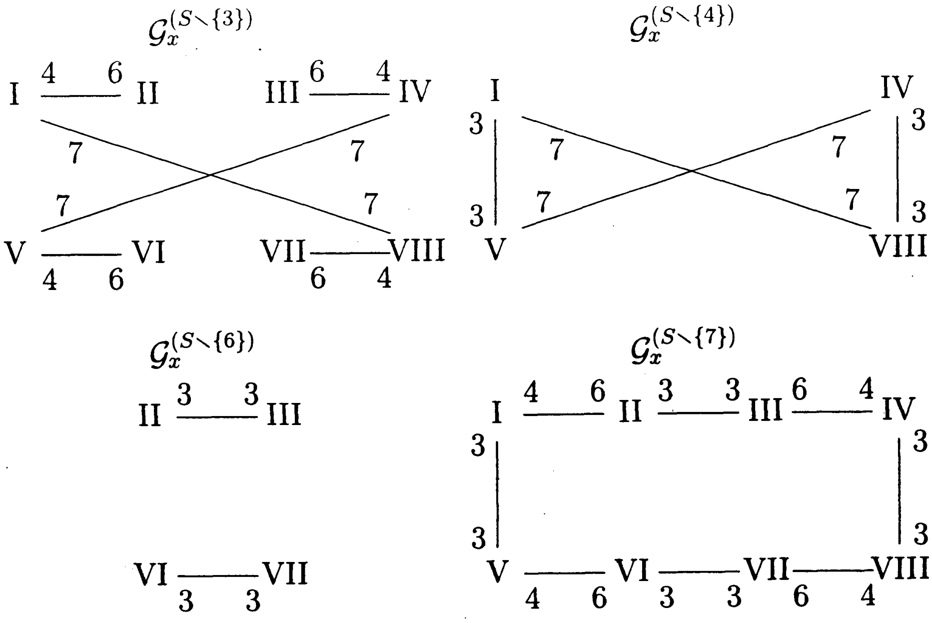


Figure 4: Subgraphs $\mathcal{G}_x^{(J)}$ of Figure 3

we have:

$$\begin{aligned} H_{x,x} &\simeq^o \left\langle \mathcal{E}_x \mid c_1 = 1, c_2 = 1, ee^{-1} = 1 \ (e \in \mathcal{E}_x), e = 1 \ (e \in T) \right\rangle \\ &\simeq \langle e(\mathrm{I},\mathrm{VIII}), e(\mathrm{IV},\mathrm{V}), e(\mathrm{VIII},\mathrm{VII}) \mid e(\mathrm{I},\mathrm{VIII})e(\mathrm{IV},\mathrm{V}) = 1, e(\mathrm{VIII},\mathrm{VII}) = 1 \rangle \\ &\simeq \langle e(\mathrm{IV},\mathrm{V}) \mid \ \rangle \simeq \mathbb{Z}. \end{aligned}$$

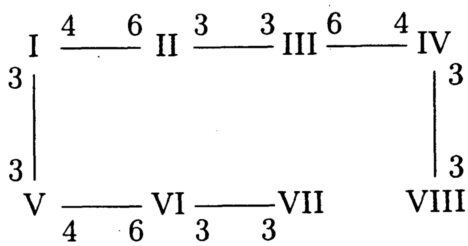


Figure 5: Maximal tree in Figure 3

The corresponding anti-isomorphism sends $e(\mathrm{IV},\mathrm{V})$ to $F(q) \in H_{x,x}$, where

$$q = e(\mathrm{II},\mathrm{III})e(\mathrm{III},\mathrm{IV})e(\mathrm{IV},\mathrm{V})e(\mathrm{V},\mathrm{I})e(\mathrm{I},\mathrm{II}) = e_{\mathrm{II}}^3 e_{\mathrm{III}}^6 e_{\mathrm{IV}}^7 e_{\mathrm{V}}^3 e_{\mathrm{I}}^4,$$

so $H_{x,x}$ is the free group generated by $F(q)$.

4. We describe the structure of $G_{x,x}$ as in Section 5. Firstly, it follows from Figure 2 that $\mathrm{CO}(x) = \{\{1\},\{2\},\{3,4\},\{5\}\}$, $\mathrm{CO}^{\geq 1}_{<\infty}(x) = \{\{3,4\}\}$. Put $\mathcal{A}_0 = \{\{3,4\}\}$, then $x^{\mathcal{A}_0} = (1,2,5,4,8) = \mathrm{VII}$ and so $E_x = \{\emptyset, \mathcal{A}_0\}$. Let $p_\emptyset \in \mathcal{P}_{x,x}$ be the trivial path and let

$$p_{\mathcal{A}_0} = e(\mathrm{II},\mathrm{I})e(\mathrm{I},\mathrm{V})e(\mathrm{V},\mathrm{VI})e(\mathrm{VI},\mathrm{VII}) = e^6_{\mathrm{II}}e^3_{\mathrm{I}}e^4_{\mathrm{V}}e^3_{\mathrm{VI}} \in \mathcal{P}_{x,x^{\mathcal{A}_0}}.$$

Then Theorem 5.7 implies $G_{x,x} = H_{x,x} \sqcup aH_{x,x}$, where $a = w_0(\{4,5\})F(p_{\mathcal{A}_0})$. Hence $G_{x,x}$ is generated by a and $F(q)$.

As remarked in the last of Section 5, $\rho_{\mathcal{A}_0}(p_{\mathcal{A}_0})$ is the path which starts at $x^{\mathcal{A}_0} = \mathrm{VII}$ and traces the directed edges labeled as $6,3,4,3$ one by one; that is,

$$\rho_{\mathcal{A}_0}(p_{\mathcal{A}_0}) = e(\mathrm{VII},\mathrm{VIII})e(\mathrm{VIII},\mathrm{IV})e(\mathrm{IV},\mathrm{III})e(\mathrm{III},\mathrm{II}).$$

We write $p \sim_F p'$ for two paths p, p' if $F(p) = F(p')$. Then we have

$$\begin{aligned}
&p_{\mathcal{A}_0}\rho_{\mathcal{A}_0}(p_{\mathcal{A}_0})\\
&= e(\mathrm{II},\mathrm{I})e(\mathrm{I},\mathrm{V})e(\mathrm{V},\mathrm{VI})e(\mathrm{VI},\mathrm{VII})e(\mathrm{VII},\mathrm{VIII})e(\mathrm{VIII},\mathrm{IV})e(\mathrm{IV},\mathrm{III})e(\mathrm{III},\mathrm{II})\\
&\sim e(\mathrm{II},\mathrm{I})c_2{}^{-1}e(\mathrm{II},\mathrm{I})^{-1} \sim_F 1
\end{aligned}$$

since $F(c_2) = 1$. So we have $a^2 = F(p_{\mathcal{A}_0}\rho_{\mathcal{A}_0}(p_{\mathcal{A}_0})) = 1$ by Corollary 5.11 (ii). Similarly, we have $\rho_{\mathcal{A}_0}(q) = e(\mathrm{VII},\mathrm{VI})e(\mathrm{VI},\mathrm{V})e(\mathrm{V},\mathrm{IV})e(\mathrm{IV},\mathrm{VIII})e(\mathrm{VIII},\mathrm{VII})$ and so

$$\begin{aligned}
&\rho_{\mathcal{A}_0}((p_{\mathcal{A}_0})^{-1}qp_{\mathcal{A}_0})\\
&= e(\mathrm{II},\mathrm{III})e(\mathrm{III},\mathrm{IV})e(\mathrm{IV},\mathrm{VIII})e(\mathrm{VIII},\mathrm{VII})\\
&\qquad \cdot e(\mathrm{VII},\mathrm{VI})e(\mathrm{VI},\mathrm{V})e(\mathrm{V},\mathrm{IV})e(\mathrm{IV},\mathrm{VIII})e(\mathrm{VIII},\mathrm{VII})\\
&\qquad \cdot e(\mathrm{VII},\mathrm{VIII})e(\mathrm{VIII},\mathrm{IV})e(\mathrm{IV},\mathrm{III})e(\mathrm{III},\mathrm{II})\\
&\sim \big(e(\mathrm{II},\mathrm{III})e(\mathrm{III},\mathrm{IV})e(\mathrm{IV},\mathrm{VIII})e(\mathrm{VIII},\mathrm{VII})e(\mathrm{VII},\mathrm{VI})e(\mathrm{VI},\mathrm{V})\big)e(\mathrm{V},\mathrm{IV})e(\mathrm{IV},\mathrm{III})e(\mathrm{III},\mathrm{II})\\
&\sim_F \big(e(\mathrm{II},\mathrm{I})e(\mathrm{I},\mathrm{V})\big)e(\mathrm{V},\mathrm{IV})e(\mathrm{IV},\mathrm{III})e(\mathrm{III},\mathrm{II}) \quad (F(c_2) = 1)\\
&= q^{-1}.
\end{aligned}$$

Hence we have $aF(q)a^{-1} = F(\rho_{\mathcal{A}_0}(p_{\mathcal{A}_0}^{-1}qp_{\mathcal{A}_0})) = F(q)^{-1}$ by Corollary 5.11 (i).

By these calculation, $\{1, a\}$ forms a subgroup of $G_{x,x}$ isomorphic to E_x, and we have

$$G_{x,x} \simeq H_{x,x} \rtimes E_x \simeq \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}$ as multiplication by -1. Moreover, put $a' = a$ and $b' = aF(q)$, then we have $G_{x,x} = \langle a', b' \mid a'^2 = 1, b'^2 = 1\rangle \simeq \widetilde{A_1}$.

5. Finally, we describe the action of $G_{x,x}$ on $W_x^\perp$. By direct computation, we have

$$a' \cdot \beta = \beta', \ b' \cdot \beta = \beta.$$

Now recall (Theorem 3.9) that both a', b' act on $W_x^\perp$ as automorphisms of the Coxeter graph $\widetilde{\Gamma}$ of $(W_x^\perp, \widetilde{S}_x)$; so we have $a' \cdot \beta' = \beta$, $b' \cdot \beta' = \beta'$.

Summarizing, we have $C_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^3 \times (\widetilde{A_1} \rtimes \widetilde{A_1})$, where each of two generators of the right $\widetilde{A_1}$ acts on the left $\widetilde{A_1}$ trivially, as the unique involution of $\widetilde{\Gamma}$ respectively.

In this example, $G_{x,x}$ is isomorphic to the semidirect product of $H_{x,x}$ by E_x, and $H_{x,x}$ forms a free group. But these properties may fail in general.

Let (W, S) be as in Figure 6 and let $x = (1, 2, 4, 5, 7, 8)$. Then it can be proved that

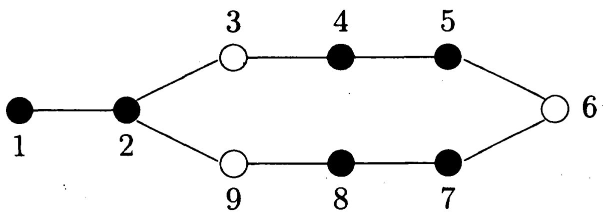


Figure 6: Coxeter graph of another example

$$W_{[x]^{\mathrm{iso}}} = 1, \ W_x^\perp = 1, \ G_{x,x} \simeq \mathbb{Z}^2, \ H_{x,x} \simeq (2\mathbb{Z})^2.$$

Thus $H_{x,x}$ is not a free group, and $G_{x,x}$ is not isomorphic to a semidirect product of $H_{x,x}$ by any group, since $G_{x,x}$ has no subgroup isomorphic to $G_{x,x}/H_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

Finally, we consider the centralizers of maximal parabolic subgroups (that is, parabolic subgroups generated by maximal proper subsets of S). Note that for $I \subset S$, the centralizer $C_W(W_I)$ of W_I is the direct product of $C_{W_{S_i}}(W_{I \cap S_i})$, where S_i runs over all connected components of S. Thus we assume that S is (finite and) connected.

Let I be a maximal proper subset of S, with connected components $I_1, \dots, I_k$. For $J \subset S$ such that J is of finite type and $\sigma_J = \mathrm{id}_J$, let $-1_J = w_0(J)$. Then it is obvious that each -1_{I_j}, -1_S is contained in $C_W(W_I)$ whenever it exists. Conversely, it can be deduced, by using the result of this paper, that $C_W(W_I)$ is generated by these elements for almost all (possibly infinite) W and I, except only two cases.

One of the exception is the case $W = D_{2n+1}$, $n \geqslant 2$ and $I = S \smallsetminus \{s_{2i}\}$, $1 \leqslant i \leqslant n-1$ (we use the numbering on S in Figure 1); in this case, $C_W(W_I)$ is generated by the involution $w_0(S)w_0(I')$, where $I' = \{s_{2i+1}, s_{2i+2}, \dots, s_{2n+1}\}$ is a connected component of I. The other is the case $W = E_6$ and $I = S \smallsetminus \{s_2\}$; now $C_W(W_I)$ is generated by the involution $w_0(S)w_0(I)$.

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